

Hierarchy for groups acting on hyperbolic \mathbb{Z}^n -spaces

Andrei-Paul Grecianu

Alexei Myasnikov

Denis Serbin

Abstract

In [7], the authors introduced the notion of hyperbolic Λ -metric spaces, which are metric spaces which the metric taking values in an ordered abelian group Λ . The present article concentrates on the case $\Lambda = \mathbb{Z}^n$ equipped with the right lexicographic order and studies the structure of finitely generated groups acting on hyperbolic \mathbb{Z}^n -metric spaces. Under certain constraints, the structure of such groups is described in terms of a *hierarchy* (see [17]) similar to the one established for \mathbb{Z}^n -free groups in [8].

1 Introduction

In his paper [10], Lyndon studied groups, where many standard cancellation techniques from the theory of free groups would successfully work too. This is how groups with (Lyndon) length functions were introduced. Then in [2], Chiswell showed that a group with a Lyndon length function taking values in \mathbb{R} (or \mathbb{Z}) has an isometric action on an \mathbb{R} -tree (or \mathbb{Z} -tree), providing a construction of the tree on which the group acts. This result was later generalized by Alperin and Bass in [1] to the case of an arbitrary ordered abelian group Λ . Hence, one can study groups with abstract Lyndon length functions taking values in Λ by considering corresponding actions on Λ -trees, the objects introduced by Morgan and Shalen in [12]. One can think of Λ -trees as 0-hyperbolic metric spaces, where the metric takes values not in \mathbb{R} but rather in the ordered abelian group Λ .

Once the equivalence between Λ -valued Lyndon length functions and actions on Λ -trees is established, one can think of possible generalizations. For example what happens if a group acts on a hyperbolic Λ -metric space? Is there an underlying length function with values in Λ in this case? This question was positively answered in [7], where the authors introduced hyperbolic Λ -valued length functions and studied many properties of groups which admit such functions.

Now let us concentrate on the case when $\Lambda = \mathbb{Z}^n$ equipped with the right lexicographic order. Our goal is to obtain the structure of groups with hyperbolic \mathbb{Z}^n -valued length functions (that is, acting on hyperbolic \mathbb{Z}^n -metric spaces) in terms of free constructions. In this paper we made the first step in this direction: under several natural restrictions on the group and the underlying hyperbolic \mathbb{Z}^n -valued length function, we obtained a result similar to the description of

groups with free regular length functions in \mathbb{Z}^n established in [8]. In the future we hope to generalize our result by dropping some of the imposed conditions.

In order to be able to precisely formulate our results below, we need several definitions (see Section 2 for details).

Let Λ be an ordered abelian group. A Λ -metric space is a space defined by the same set of axioms as a usual metric space but with \mathbb{R} replaced by Λ . Now, if (X, d) is a Λ -metric space, $v \in X$ and $\delta \in \Lambda$ is positive, then we call (X, d) δ -hyperbolic with respect to v if for all $x, y, z \in X$

$$(x \cdot y)_v \geq \min\{(x \cdot z)_v, (z \cdot y)_v\} - \delta.$$

In fact, hyperbolicity does not depend on the choice of the point v , that is, a δ -hyperbolic space with respect to one point is 2δ -hyperbolic with respect to any other point. So, we call (X, d) δ -hyperbolic (or simply *hyperbolic*) if it is δ -hyperbolic with respect to every $v \in X$.

Now suppose a group G is acting on a Λ -metric space (X, d) . If we fix $x_0 \in X$ then the action $G \curvearrowright X$ provides a length function $l : G \rightarrow \Lambda$ defined by $l(g) = d(x, gx_0)$. If (X, d) is hyperbolic then the length function l defined above is also called *hyperbolic* and it satisfies the set of axioms $(\Lambda 1)$ – $(\Lambda 4)$ (see Section 2 for details). Note that one can consider an abstract function $l : G \rightarrow \Lambda$ on G satisfying the axioms $(\Lambda 1)$ – $(\Lambda 4)$, that is, without underlying action on any space. In this case we still call such a function hyperbolic.

Next, that l is an abstract hyperbolic length function on G , we write $ab = a \circ b$ to signify that $l(ab) = l(a) + l(b)$ and we say that the function l is δ -regular if for any $g, h \in G$ there exist g_c, h_c, g_d, h_d such that

$$l(g_c) = l(h_c) = c(g, h), \quad g = g_c \circ g_d, \quad h = h_c \circ h_d, \quad \text{and} \quad l(g_c^{-1}h_c) \leq 4\delta,$$

where

$$c(g, h) = \frac{1}{2} (l(g) + l(h) - l(g^{-1}h)).$$

The notion of regularity of length function (for groups acting on Λ -trees) first appeared in [15] and then was developed in [13]. Regularity of Λ -valued hyperbolic length functions was studied in [7] (see Section 5.1).

Now consider the case $\Lambda = \mathbb{Z}^n$, where \mathbb{Z}^n has the right lexicographic order. Such an order gives rise to the notion of *height*: for $a \in \mathbb{Z}^n$ we define the height $ht(a)$ of a to be equal to k if $a = (a_1, \dots, a_k, 0, \dots, 0)$ and $a_k \neq 0$. In particular, for $a, b > 0$ the inequality $ht(a) < ht(b)$ implies $a < b$.

Let G be a group and let $l : G \rightarrow \mathbb{Z}^n$ be an abstract length function. We say l is *proper* if for any $k \in \mathbb{N}$ the set $\{g \mid l(g) \leq (k, 0, \dots, 0)\}$ is finite. That is, the set of elements of G whose lengths are bounded by any natural number (considered as a subset of \mathbb{Z}^n) is finite if the length function is proper. For example, if $l : G \rightarrow \mathbb{Z}$ is the word length function associated with a finite generating set of G then l is obviously proper. But if l takes values in \mathbb{Z}^n for $n > 1$ then it well may be the case that there are infinitely many group elements of a fixed finite length $k \in \mathbb{N}$ is infinite.

Now we are ready to formulate the main result of the paper. Recall that a subgroup H of a group G is called *isolated* if whenever there exists some $n \in \mathbb{Z}$ such that $g^n \in H$, it implies that $g \in H$.

Theorem 3.1.1. *Let G be a finitely generated torsion-free group and let $l : G \rightarrow \mathbb{Z}^n$, where \mathbb{Z}^n has the right lexicographic order, be a proper, δ -regular, δ -hyperbolic length function with the following restrictions:*

- (a) $ht(\delta) = 1$,
- (b) $\{g \in G \mid ht(l(g)) = 1\}$ is a finitely generated isolated subgroup of G .

Then G can be represented as a union of a finite series of groups

$$G_1 < G_2 < \cdots < G_n = G,$$

where G_1 is a word hyperbolic group and for every $k < n$, the group G_{k+1} is isomorphic to an HNN extension of G_k with a finite number of stable letters and whose associated subgroups are virtually nilpotent of rank at most 3.

Note that the condition that G is finitely generated is required to make sure that G_1 is word hyperbolic and it is sufficient in this context. There are of course non-finitely generated groups with proper regular actions on hyperbolic spaces, in which case Theorem 3.1.1 may not hold. For example, consider an infinitely generated free group $F = F(X)$, where $X = \{x_i \mid i \in \mathbb{N}\}$, acting on its Cayley graph $\Gamma(F, X)$. If all edges of $\Gamma(F, X)$ have the same length 1 then the length function arising from the action is regular but not proper. At the same time, if we define the length of every edge labeled by x_n to be n then regularity still holds, but the length function becomes proper. Observe that $F(X)$ cannot be decomposed as shown in Theorem 3.1.1: G_1 in this case coincides with $F(X)$ itself which is not hyperbolic.

Most of the results in this paper also appear in the first author's Ph.D Thesis (see [6]).

Acknowledgement. The authors would like to thank Olga Kharlampovich for insightful discussions and the referees for their thorough reviews and very useful comments, which helped immensely in improving the clarity and relevance of the paper.

2 Preliminaries

Here we recall the basic definitions regarding hyperbolic Λ -metric spaces (see [7] for details).

2.1 Λ -hyperbolic spaces and length functions

Recall that an *ordered* abelian group is an abelian group Λ (with addition denoted by “+”) equipped with a linear order “ \leq ” such that for all $\alpha, \beta, \gamma \in \Lambda$ the inequality $\alpha \leq \beta$ implies $\alpha + \gamma \leq \beta + \gamma$.

Sometimes we would like to be able to divide elements of Λ by non-zero integers. To this end we fix a canonical order-preserving embedding of Λ into an ordered *divisible* abelian group $\Lambda_{\mathbb{Q}}$ and identify Λ with its image in $\Lambda_{\mathbb{Q}}$. The group $\Lambda_{\mathbb{Q}}$ is the tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$ of two abelian groups (viewed as \mathbb{Z} -modules) over \mathbb{Z} . One can represent elements of $\Lambda_{\mathbb{Q}}$ by fractions $\frac{\lambda}{m}$, where $\lambda \in \Lambda, m \in \mathbb{Z}, m \neq 0$, and two fractions $\frac{\lambda}{m}$ and $\frac{\mu}{n}$ are equal if and only if $n\lambda = m\mu$. Addition of fractions is defined as usual, and the embedding is given by the map $\lambda \rightarrow \frac{\lambda}{1}$. The order on $\Lambda_{\mathbb{Q}}$ is defined by $\frac{\lambda}{m} \geq 0 \iff m\lambda \geq 0$ in Λ . Obviously, the embedding $\Lambda \rightarrow \Lambda_{\mathbb{Q}}$ preserves the order. It is easy to see that $\mathbb{R}_{\mathbb{Q}} = \mathbb{R}$ and $\mathbb{Z}_{\mathbb{Q}} = \mathbb{Q}$. Furthermore, it is not hard to show that $(A \oplus B)_{\mathbb{Q}} \simeq A_{\mathbb{Q}} \oplus B_{\mathbb{Q}}$, so $(\mathbb{R}^n)_{\mathbb{Q}} = \mathbb{R}^n$ and $(\mathbb{Z}^n)_{\mathbb{Q}} = \mathbb{Q}^n$. Notice also, that for every Λ the group $\mathbb{Z} \oplus \Lambda$ is discrete.

For elements $\alpha, \beta \in \Lambda$ the *closed segment* $[\alpha, \beta]$ is defined by

$$[\alpha, \beta] = \{\gamma \in \Lambda \mid \alpha \leq \gamma \leq \beta\}.$$

Now a subset $C \subset \Lambda$ is called *convex* if for every $\alpha, \beta \in C$ the set C contains $[\alpha, \beta]$. In particular, a subgroup C of Λ is convex if $[0, \beta] \subset C$ for every positive $\beta \in C$.

Let X be a non-empty set and Λ an ordered abelian group. A Λ -metric on X is a mapping $d : X \times X \rightarrow \Lambda$ such that:

- (LM1) $\forall x, y \in X : d(x, y) \geq 0$;
- (LM2) $\forall x, y \in X : d(x, y) = 0 \iff x = y$;
- (LM3) $\forall x, y \in X : d(x, y) = d(y, x)$;
- (LM4) $\forall x, y, z \in X : d(x, y) \leq d(x, z) + d(y, z)$.

A Λ -metric space is a pair (X, d) , where X is a non-empty set and d is a Λ -metric on X . If (X, d) and (X', d') are Λ -metric spaces, an *isometry* from (X, d) to (X', d') is a mapping $f : X \rightarrow X'$ such that $d(x, y) = d'(f(x), f(y))$ for all $x, y \in X$. As in the case of usual metric spaces, a *segment* in a Λ -metric space X is the image of an isometry $\alpha : [a, b] \rightarrow X$ for some $a, b \in \Lambda$. In this case $\alpha(a), \alpha(b)$ are called the endpoints of the segment. By $[x, y]$ we denote any segment with endpoints $x, y \in X$.

We call a Λ -metric space (X, d) *geodesic* if for all $x, y \in X$, there is a segment in X with endpoints x, y .

Let (X, d) be a Λ -metric space. Fix a point $v \in X$ and for $x, y \in X$ define the Gromov product

$$(x \cdot y)_v = \frac{1}{2}(d(x, v) + d(y, v) - d(x, y)),$$

as an element of $\Lambda_{\mathbb{Q}}$. Obviously, this is the direct generalization of the standard Gromov product to the Λ -metric case. Most of its classical properties remain true in the generalization (see [7, Section 2.3]).

Let $\delta \in \Lambda$ with $\delta \geq 0$. Then (X, d) is δ -hyperbolic with respect to v if, for all $x, y, z \in X$

$$(x \cdot y)_v \geq \min\{(x \cdot z)_v, (z \cdot y)_v\} - \delta.$$

It was proved in [3, Lemma 1.2.5] that if X is δ -hyperbolic with respect to x then it is 2δ -hyperbolic with respect to any other point $y \in X$. In view of this result, we call a Λ -metric space (X, d) δ -hyperbolic if it is δ -hyperbolic with respect to any point.

One of the crucial examples of δ -hyperbolic Λ -metric spaces is *Lambda-tree* which is exactly a 0-hyperbolic geodesic Λ -metric space X such that the Gromov product of any two points $x, y \in X$ with respect to any other point $v \in X$ is always an element of Λ , that is, $(x \cdot y)_v \in \Lambda$ for all $x, y, v \in X$.

In [10] Lyndon introduced a notion of an (abstract) *length function* $l : G \rightarrow \Lambda$ on a group G with values in Λ . Such a function l satisfies the following axioms:

$$(\Lambda 1) \quad \forall g \in G : l(g) \geq 0 \text{ and } l(1) = 0,$$

$$(\Lambda 2) \quad \forall g \in G : l(g) = l(g^{-1}),$$

$$(\Lambda 3) \quad \forall g, h \in G : l(gh) \leq l(g) + l(h).$$

Again, the easiest example of a \mathbb{Z} -valued length function on any group G with a generating set S is the word length $|\cdot|_S : G \rightarrow \mathbb{Z}$, where $|g|_S$ is the minimum length of a word w in the generators from S representing the element $g \in G$.

We can introduce another axiom which is a direct generalization of the axiom Lyndon introduced. A length function $l : G \rightarrow \Lambda$ is called *hyperbolic* if there is $\delta \in \Lambda$ such that

$$(\Lambda 4, \delta) \quad \forall f, g, h \in G : c(f, g) \geq \min\{c(f, h), c(g, h)\} - \delta,$$

where $c(g, h) = \frac{1}{2} (l(g) + l(h) - l(g^{-1}h))$ is viewed as an element of $\Lambda_{\mathbb{Q}}$.

If G acts on a Λ -metric space (X, d) then one can fix a point $v \in X$ and consider a function $l_v : G \rightarrow \Lambda$ defined as $l_v(g) = d(v, gv)$, called a *length function based at v* . It is not hard to show that l_v satisfies the axioms $(\Lambda 1)$ - $(\Lambda 3)$, that is, it is a length function on G with values in Λ . Moreover, if (X, d) is δ -hyperbolic for some $\delta \in \Lambda$ with respect to v then l_v is δ -hyperbolic (see [7, Theorem 3.1]).

It is easy to see that a length function $l : G \rightarrow \Lambda$ defines a Λ -pseudometric on G defined by $d_l(g, h) = l(g^{-1}h)$. In the case when $l(g) = 0$ if and only if $g = 1$ the pseudometric d_l becomes a Λ -metric.

Example 2.1.1. Let (Γ, d) be a Λ -tree and G a group acting on Γ by isometries. Let $x \in \Gamma$ have a trivial stabilizer and define $l(g) = d(x, gx)$. Noticing that

$$d(g, h) = l(g^{-1}h) = d(x, (g^{-1}h)x) = d(gx, hx)$$

and remembering that Γ is 0-hyperbolic (since it is a Λ -tree) gives us that d is a 0-hyperbolic length function on G .

Now suppose that $l : G \rightarrow \Lambda$ is a δ -hyperbolic length function. We write $ab = a \circ b$ to signify that $l(ab) = l(a) + l(b)$ and we say that the function l is δ -regular if for any $g, h \in G$ there exist g_c, h_c, g_d, h_d such that

$$l(g_c) = l(h_c) = c(g, h), \quad g = g_c \circ g_d, \quad h = h_c \circ h_d, \quad \text{and} \quad l(g_c^{-1}h_c) \leq 4\delta.$$

Compare this definition with the ones using the properties $(R1, k)$, $(R2, k)$, and $(R3, k)$ introduced in [7, Section 5.1].

The regularity property of (Lyndon) length functions was first considered in [15]. Then it was developed in [13] for groups acting freely on Λ -trees. It turns out that this property makes many combinatorial arguments based on cancellation between elements very similar to the case of free groups (for example, Nielsen method in free groups). In particular, it is possible to describe groups which have regular free length functions in \mathbb{Z}^n (see [8]) and, more generally, in an arbitrary ordered abelian group Λ (see [9]). Regularity of Λ -valued hyperbolic length functions was first introduced in [7], where basic properties of such functions were studied.

In our case, regularity of a δ -hyperbolic length function on a group G will help us determine the structure of G similarly to the case $\delta = 0$.

2.2 The case $\Lambda = \mathbb{Z}^n$

Now consider the case when $\Lambda = \mathbb{Z}^n$, where \mathbb{Z}^n has the right lexicographic order. Recall that if A and B are ordered abelian groups, then the *right lexicographic order* on the direct sum $A \oplus B$ is defined as follows:

$$(a_1, b_1) < (a_2, b_2) \Leftrightarrow b_1 < b_2 \text{ or } b_1 = b_2 \text{ and } a_1 < a_2.$$

One can easily extend this definition to any number of components in the direct sum and apply it in the case of \mathbb{Z}^n which is the direct sum of n copies of \mathbb{Z} .

Every element $a \in \mathbb{Z}^n$ can be represented by an n -tuple $(a_1, \dots, a_k, 0, \dots, 0)$. We say that the height of a is equal to k , and write $ht(a) = k$, if $a = (a_1, \dots, a_k, 0, \dots, 0)$ and $a_k \neq 0$. The notion of height can be defined for an arbitrary ordered abelian group Λ since Λ can be represented as the union of all its convex subgroups and the height of $\alpha \in \Lambda$ is the smallest index of the convex subgroup of Λ that α belongs to.

Speaking of convex subgroups of \mathbb{Z}^n , each one of them is of the following type

$$\{a \in \mathbb{Z}^n \mid ht(a) \leq k\}$$

which is naturally isomorphic to \mathbb{Z}^n . Hence, we have a (finite) complete chain of convex subgroups of \mathbb{Z}^n :

$$0 < \mathbb{Z} < \mathbb{Z}^2 < \dots < \mathbb{Z}^n.$$

Now, if (X, d) is a \mathbb{Z}^n -metric space and $C \leq \mathbb{Z}^n$ is a convex subgroup, then we can consider the subset

$$X_{x,C} = \{y \in X \mid d(x, y) \in C\}$$

which is in turn a C -metric space with respect to the metric $d_0 = d|_{X_{x,C}}$. We call $(X_{x,C}, d_0)$ a C -metric subspace of X . In view of the complete chain of convex subgroups of \mathbb{Z}^n above, we may have \mathbb{Z}^k -subspaces of X for every $k \in [1, n]$. In particular, we can say that a subspace X_0 of X has height k if $X_0 = X_{x,C}$ for some $x \in X$ and $C = \mathbb{Z}^k$.

Let G act on a \mathbb{Z}^n -metric space (X, d) . We say that the action is *proper* if for any $x \in X$, the intersection $(G \cdot x) \cap B_\kappa(x)$ is finite for every $\kappa = (a, 0, \dots, 0)$, where $a \in \mathbb{N}$ (here $G \cdot x$ is the orbit of the point $x \in X$ and $B_\kappa(x)$ is a ball of radius κ in X centered at x). We say that the action is *co-compact* if there exists $\kappa = (a, 0, \dots, 0)$, where $a \in \mathbb{N}$, such that for every $x, y \in X$ there exist $x' \in G \cdot x$, $y' \in G \cdot y$ with the property $d(x', y') \leq \kappa$.

Now, assume that a group G has a δ -hyperbolic length function l with values in \mathbb{Z}^n . In many cases we will refer to the height $ht(g)$ of $g \in G$ which is simply the height of its length $ht(l(g))$.

One of the first examples of \mathbb{Z}^n -valued hyperbolic length functions is the function $l : G \rightarrow \mathbb{Z}$, where G is a word-hyperbolic group with a generating set S and $l = |\cdot|_S$ is the word length with respect to S . There are more sophisticated examples.

Example 2.2.1. Let G be a group acting on a hyperbolic \mathbb{Z} -metric space X and a \mathbb{Z} -tree Y . Let $x \in X$ and $y \in Y$.

From the actions of G on X and Y we obtain the length functions $l_X : G \rightarrow \mathbb{Z}$ and $l_Y : G \rightarrow \mathbb{Z}$ based respectively at x and y , that is, defined as $l_X(g) = d(x, gx)$ and $l_Y(g) = d(y, gy)$. Next, we have

$$c_X(g, h) = \frac{1}{2} (l_X(g) + l_X(h) - l_X(g^{-1}h))$$

and

$$c_Y(g, h) = \frac{1}{2} (l_Y(g) + l_Y(h) - l_Y(g^{-1}h)).$$

Define $l : G \rightarrow \mathbb{Z}^2$ as $l = (l_X, l_Y)$, that is,

$$l(g) = (d(x, gx), d(y, gy)).$$

Hence, we have

$$c(g, h) = \frac{1}{2} (l(g) + l(h) - l(g^{-1}h)) = (c_X(g, h), c_Y(g, h)).$$

This implies that, for any $g, h, k \in G$, we have

$$\begin{aligned} c(g, k) &= (c_X(g, k), c_Y(g, k)) \\ &\geq (\max\{c_X(g, h), c_X(h, k)\} - \delta, \max\{c_Y(g, h), c_Y(h, k)\}) \\ &= \max\{c(g, k), c(h, k)\} - (\delta, 0). \end{aligned}$$

This implies that any group that has compatible actions on a simplicial tree and a hyperbolic \mathbb{Z} -metric space has a \mathbb{Z}^2 -valued hyperbolic length function. The same construction is valid for Y being a \mathbb{Z}^n -tree.

Let $l : G \rightarrow \mathbb{Z}^n$ be δ -hyperbolic. We say l is *proper* if for any $k \in \mathbb{N}$ the set $\{g \mid l(g) \leq (k, 0, \dots, 0)\}$ is finite. Observe that if l is a based length function coming from a proper action of G on some \mathbb{Z}^n -metric space X then l itself is proper. A proper length function is simply one that corresponds to an action that is properly discontinuous on subspaces of height 1, hence the term.

Example 2.2.2. *Let a finitely generated group G act properly and co-compactly on a geodesic \mathbb{Z}^n -hyperbolic space X so that the stabilizer of some \mathbb{Z} -subspace X_0 is finitely generated.*

Since G acts co-compactly, X/G has finite diameter of minimal height, which means that there exists $a > 0$ such that for any $x, y \in X$ there exists $g \in G$ with the property $gx \in B_\kappa(y)$, where $\kappa = (a, 0, \dots, 0)$. This naturally implies that the action of G satisfies the condition $(RA, 1)$ (see [7, Section 5.1]), since there is always a mid-point y of the geodesic triangle $\{x, gx, hx\}$ and there exists some $u \in G$ such that $ux \in B_\kappa(y)$.

Take then $x_0 \in X_0$ and define $l(g) = d(x_0, gx_0)$. Note that l is hyperbolic since X is hyperbolic. Moreover, it is regular which follows from the fact that the action satisfies the condition $(RA, 1)$ (see [7, Lemma 5.3]).

It follows that natural conditions (proper and co-compact action) on a geodesic \mathbb{Z}^n -hyperbolic space ensures existence of a regular and proper \mathbb{Z}^n -valued length function.

3 The structure of groups with \mathbb{Z}^n -valued hyperbolic length functions

In this section we state and prove the main result of the paper.

3.1 The main theorem

The goal of this section is to prove the following theorem.

Theorem 3.1.1. *Let G be a finitely generated torsion-free group and let $l : G \rightarrow \mathbb{Z}^n$, where \mathbb{Z}^n has the right lexicographic order, be a proper, δ -regular, δ -hyperbolic length function with the following restrictions:*

- (a) $ht(\delta) = 1$,
- (b) $\{g \in G \mid ht(l(g)) = 1\}$ is a finitely generated isolated subgroup of G .

Then G can be represented as a union of a finite series of groups

$$G_1 < G_2 < \dots < G_n = G,$$

where G_1 is a word hyperbolic group and for every $k < n$, the group G_{k+1} is isomorphic to an HNN extension of G_k with a finite number of stable letters and whose associated subgroups are virtually nilpotent of rank at most 3.

Recall Example 2.2.2 above. Note that the condition (b) of Theorem 3.1.1 is satisfied by the choice of x_0 . It follows that every finitely generated group G acting properly and co-compactly on a geodesic \mathbb{Z}^n -hyperbolic space X so that the stabilizer of some \mathbb{Z} -subspace X_0 is finitely generated, by Theorem 3.1.1 can be represented as a finite series of HNN extensions described above. In light of this observation, save for one technical assumption, we can think of groups satisfying the conditions of Theorem 3.1.1 as a higher-dimensional analog of classical hyperbolic groups.

In order to prove Theorem 3.1.1 we will need an auxiliary result which has a similar statement. Recall (see [3, Chapter 3.2]) that if a group G acts on a Λ -tree T , we say the action is *abelian* if for any $g, h \in G$ we have that

$$\min_{x \in T} \{d(x, (gh)x)\} \leq \min_{x \in T} \{d(x, gx)\} + \min_{x \in T} \{d(x, hx)\}.$$

The auxiliary result mentioned above is the following proposition.

Proposition 3.1.1. *Let G be a torsion-free group and let $l : G \rightarrow \mathbb{Z}^n$, where \mathbb{Z}^n has the right lexicographic order, be a δ -regular and δ -hyperbolic length function with the following restrictions:*

- (a) $ht(\delta) = 1$,
- (b) for any $g \in G$ and $k \neq 0$, if $l(g^k) < l(g)$, then $ht(l(g) - l(g^k)) = 1$.
- (c) for every non-trivial $g \in G$ we have $l(g) > 0$.

Then G can be represented as a union of a finite series of groups

$$G_1 < G_2 < \cdots < G_n = G,$$

where G_{k+1} is isomorphic to an HNN extension of G_k , whose associated subgroups have abelian actions on a \mathbb{Z}^k -tree.

The proof of Proposition 3.1.1 is going to follow from a series of lemmas presented in the next section.

3.2 Proof of the auxiliary proposition

Throughout this section, all lemmas assume the same preconditions as Proposition 3.1.1. That is, G is a torsion-free group, $l : G \rightarrow \mathbb{Z}^n$, where \mathbb{Z}^n has the right lexicographic order, is a δ -regular and δ -hyperbolic length function such that:

- (A) $ht(\delta) = 1$,
- (B) for any $g \in G$ and $k \neq 0$, if $l(g^k) < l(g)$, then $ht(l(g) - l(g^k)) = 1$.
- (C) for every non-trivial $g \in G$ we have $l(g) > 0$.

Let $G_k = \{g \in G \mid ht(g) \leq k\}$ and let l_k be the restriction of l on the $n - k$ rightmost coordinates of $l(g)$ ($l = l_0$). Note that l_k is a \mathbb{Z}^{n-k} -valued length function on G (the axioms $(\Lambda 1)$ - $(\Lambda 3)$ are trivially satisfied). For every $f, g \in G$ denote

$$c(g, h) = \frac{1}{2} (l(g) + l(h) - l(g^{-1}h))$$

and

$$c_k(g, h) = \frac{1}{2} (l_k(g) + l_k(h) - l_k(g^{-1}h)).$$

Lemma 3.2.1. *For any $k > 0$, the length function l_k satisfies the following conditions:*

- l_k is 0-hyperbolic,
- l_k is 0-regular,
- $l_k(g^2) \geq l_k(g)$ for any $g \in G$.

Proof. Obviously, $l_k(1) = 0$ and $l_k(g) = l_k(g^{-1})$. Next, $c_k = \pi_k \circ c$, where π_k is the projection on the $n - k$ rightmost coordinates. Note that π_k is an order-preserving homomorphism of \mathbb{Z}^n onto \mathbb{Z}^{n-k} . Therefore, since $c(g, h) \geq \min\{c(g, k), c(h, k)\} - \delta$ for any g, h and k , we have that

$$\begin{aligned} c_k(g, h) &= \pi_k(c(g, h)) \geq \min\{\pi_k(c(g, k)), \pi_k(c(h, k))\} - \pi_k(\delta) \\ &= \min\{c_k(g, k), c_k(h, k)\} \end{aligned}$$

since $\pi_k(\delta) = 0$ for any $k > 0$ (it follows from the fact that $ht(\delta) = 1$).

The condition of 0-regularity and $l_k(g^2) \geq l_k(g)$ are immediate from the fact that $l_k = \pi_k \circ l$ together with δ -regularity of l and the assumption (B) above. \square

Lemma 3.2.2. *For any $k > 0$, there exists a tree Γ whose vertices are labeled by the cosets of G_{k+1} by G_k and on which G_{k+1} acts without inversions by left multiplication.*

Proof. By Lemma 3.2.1, restricting l_k to G_{k+1} gives us a 0-hyperbolic integer length function. By [3, Theorem 4.4], it follows that G_{k+1} acts on a \mathbb{Z} -tree (or simplicial tree) Γ' . It follows that G_{k+1} acts without inversions on the barycentric subdivision of Γ' .

To construct Γ , let us look more closely at the construction of Γ' . Defining $d_k(g, h) = l_k(g^{-1}h)$ and $g \sim h$ if $d_k(g, h) = 0$, let us first take the elements of G_{k+1}/\sim and, choosing the equivalence class of the identity as basepoint, we then add vertices to form paths of length $l_k(g)$ from the equivalence class of the identity to that of g and finally glue the paths towards g and h together up to the points on $[1, g]$ and $[1, h]$ at distance $c_k(g, h)$ from 1. Label all vertices that were originally equivalence classes of G_{k+1}/\sim by an element whose equivalence class they represented. Notice that by δ -regularity, there exist g_1, g_2, h_1 and h_2 such that $g = g_1 \circ g_2$, $h = h_1 \circ h_2$, $c(g, h) = l(g_1) = l(h_1)$ and $l(g_1^{-1}h_1) < 4\delta$. It

implies that $g_1 \sim h_1$ and the branching point is the equivalence class of g_1 , so all branching points of Γ' are labeled by elements of G_{k+1}/\sim . Furthermore,

$$g \sim h \Leftrightarrow g^{-1}h \in G_k \Leftrightarrow h \in gG_k \Leftrightarrow hG_k = gG_k$$

so $G_{k+1}/\sim = G_{k+1}/G_k$, and since the action of g on the equivalence class of h is the equivalence class of gh , we have that G_{k+1} acts on the labeled vertices of Γ' by left multiplication of cosets.

Let us therefore define Γ as follows: $V(\Gamma) = G_{k+1}/G_k$, and defining $[gG_k, hG_k]$ to be the geodesic in Γ' between gG_k and hG_k , $(gG_k, hG_k) \in E(\Gamma)$ if and only if $[gG_k, hG_k]$ contains no labeled vertices except gG_k and hG_k . Since the set of labeled vertices of Γ' is closed on the action of G_k , we have that G_k acts isometrically on Γ by left multiplication.

It remains to be proven that G_{k+1} acts on Γ without inversions. Suppose, on the contrary, that there exist g, h_1 and h_2 ($h_1 \neq h_2$) such that $gh_1G_k = h_2G_k$ and $gh_2G_k = h_1G_k$ with $(h_1G_k, h_2G_k) \in E(\Gamma)$. Since G_{k+1} acts without inversions on the barycentric subdivision of Γ' , we have that $[h_1G_k, h_2G_k]$ contains a mid-point that will be a unlabeled vertex, say v .

Observe that either h_1G_k or h_2G_k , but not both belong to $[G_k, v]$. Suppose, without loss of generality, that h_1G_k belongs to $[G_k, v]$. It follows that h_2G_k is in $[gG_k, v]$ and h_1G_k is in $[g^2G_k, v]$. Therefore,

$$[G_k, gG_k] = [G_k, v] \cup [v, gG_k] = [G_k, h_1G_k] \cup [h_1G_k, h_2G_k] \cup [h_2G_k, gG_k]$$

and

$$[G_k, g^2G_k] \subset [G_k, h_1G_k] \cup [h_1G_k, g^2G_k].$$

However, this implies that $d(G_k, h_1G_k) = d(gG_k, h_2G_k) = d(g^2G_k, h_1G_k)$, since all are images of the same path after translation by g and g^2 . But that means that $d(G_k, g^2G_k) = l_k(g^2) < l_k(g) = d(G_k, gG_k)$.

This is a contradiction with our assumption that if $l(g^k) < l(g)$ then $ht(l(g) - l(g^k)) = 1$ since $d(G_k, g^2G_k) < d(G_k, gG_k) \Rightarrow ht(l(g) - l(g^2)) > k$.

Note that the same argument shows that G acts without inversions not just on the barycentric subdivision of Γ' , but on Γ' itself. \square

Proposition 3.2.1. *For any $k > 0$, there exist collections $\{C_i \mid i \in I\}$, $\{D_i \mid i \in I\}$, not necessarily finite, of subgroups of G_k and isomorphisms $\{\phi_i : C_i \rightarrow D_i \mid i \in I\}$ such that*

$$G_{k+1} = \langle G_k, \{h_i \mid i \in I\} \mid h_i^{-1}c_i h_i = \phi_i(c_i), c_i \in C_i \rangle.$$

Moreover, $C_i = G_k \cap h_i G_k h_i^{-1}$ and $D_i = G_k \cap h_i^{-1} G_k h_i$.

Proof. By lemma 3.2.2, G_{k+1} acts without inversions on a tree whose vertices are the cosets of G_k . Since G_{k+1} acts transitively on cosets, we know G_{k+1} will admit a splitting as an HNN extension with multiple stable letters.

We must therefore look at stabilizers of vertices in Γ . Let g be in G_{k+1} and h in $G_{k+1} \setminus G_k$.

$$ghG_k = hG_k \Leftrightarrow h^{-1}gh \in G_k \Leftrightarrow g \in hG_k h^{-1},$$

so the stabilizer of hG_k is $hG_k h^{-1}$. We know all loops about the unique vertex of Γ/G_{k+1} are images of edges from G_k to some hG_k , and the stabilizer of G_k is itself, so the stabilizer of $[G_k, hG_k]$ is $G_k \cap hG_k h^{-1}$.

This group will be embedded directly into both G_k and $hG_k h^{-1}$, and the latter will be sent to the basepoint via conjugation by h , so we have that the edge groups in Γ are $G_k \cap hG_k h^{-1}$, embedded directly into G_k into one direction and sent to $G_k \cap h^{-1}G_k h$ and then embedded directly into G_k in the other, giving us the splitting defined above. \square

Corollary 3.2.1. *If G_{k+1} is finitely generated, then G_{k+1} is an HNN extension of G_k with finitely many stable letters.*

Proof. Observe that G_{k+1} acts without inversions on the tree Γ and Γ/G_{k+1} is a bouquet of loops used to obtain the HNN extension (see proof of Proposition 3.2.1). By known result (Corollary 1 of [16, Theorem 5.4.13]), there exists a normal subgroup of G_{k+1} , N , such that $G_{k+1}/N \cong \pi_1(\Gamma/G_{k+1})$. Γ/G_{k+1} being a bouquet of loops, $\pi_1(\Gamma/G_{k+1}) \cong F_\nu$ where ν is the cardinality of the set of loops of Γ/G_{k+1} .

However, if $\{x_1, \dots, x_n\}$ is a (finite) generating set of G_{k+1} , then the set $\{x_1 N, \dots, x_n N\}$ generates G_{k+1}/N , so $\nu \leq n$ and there are at most n loops in Γ/G_{k+1} , and so at most n subgroups C_i used to construct G_{k+1} from G_k . \square

Definition 3.2.1. *By an argument similar to that of Lemma 3.2.1, for any $i < j$ we have that (G_j, l_i) is 0-hyperbolic. Using again Chiswell's construction of a tree from a partial 0-hyperbolic metric space, we refer to the \mathbb{Z}^{j-i} -tree built starting from (G_j, l_i) as Γ_i^j .*

Lemma 3.2.3. *Let C_i be one of the subgroups used in constructing G_{k+1} as an HNN extension of G_k , $c \in C_i$ and h_i be the stable letter associated to C_i ; then either $l_1(c) = c_1(c, h_i) + c_1(c, c^{-1})$, or $l_1(c) = c_1(c^{-1}, h_i) + c_1(c, c^{-1})$.*

Proof. $c \in C_i = G_k \cap h_i G_k h_i^{-1}$, so, in particular, $h_i^{-1} c h_i \in G_k$ and $l(h_i^{-1} c h_i) \in \mathbb{Z}^k$. Next,

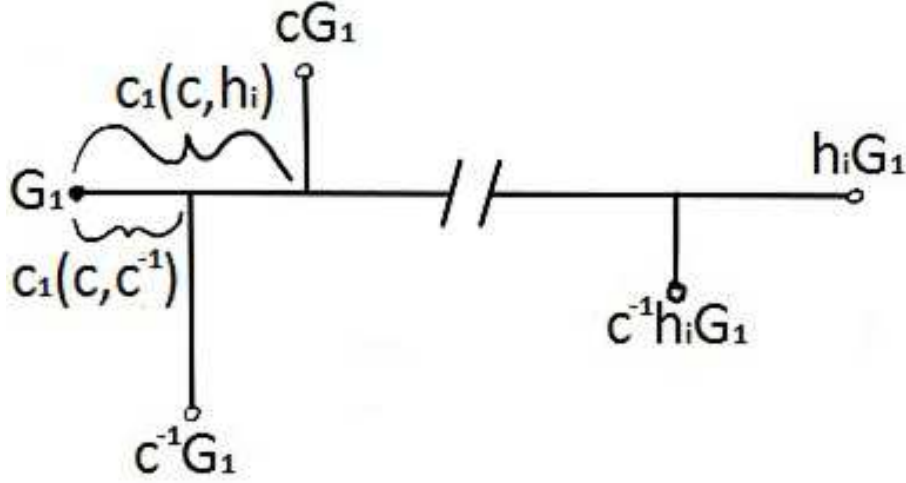
$$l(h_i^{-1} c h_i) = 2l(h_i) + l(c) - 2c(h_i, ch_i) - 2c(c^{-1}, h_i).$$

Now, $c(c^{-1}, h_i) \in \mathbb{Z}^k$, whereas $l(h_i) \in \mathbb{Z}^{k+1}$, so we know that $c(h_i, ch_i) \in \mathbb{Z}^{k+1}$, therefore we have that

$$\begin{aligned} 2c_1(c, h_i) &\geq 2 \min\{c_1(c, ch_i), c_1(ch_i, h_i)\} = 2c_1(c, ch_i) = l_1(c) + l_1(ch_i) - l_1(h_i) \\ &= 2l_1(c) - 2c_1(c^{-1}, h_i) \Rightarrow l_1(c) \leq c_1(c, h_i) + c_1(c^{-1}, h_i) \end{aligned}$$

It follows that either $c_1(c, h_i)$ or $c_1(c^{-1}, h_i)$ must be greater than or equal to $l_1(c)/2$. Suppose without loss of generality that it is the former. We also know the common initial segment of h_i and $c^{-1}h_i$ is of greater height than c since c^{-1} also stabilizes $h_i G_k$.

In Γ_1^n , the triangles $\Delta_1 = \{cG_1, G_1, h_i G_1\}$ and $\Delta_2 = \{G_1, c^{-1}G_1, c^{-1}h_i G_1\}$ are isometric (translation by c). The branch point of Δ_1 is on the segment



$[G_1, cG_1]$ at distance $c_1(c, h_i)$ from the point G_1 it follows that the branch point of Δ_2 will be on the segment $[c^{-1}G_1, G_1]$ at distance $c_1(c, h_i)$ from the point $c^{-1}G_1$. Since $l_1(c^2) \geq l_1(c)$, we also know $c_1(c, c^{-1}) \leq l_1(c)/2 \leq c_1(c, h_i) \leq c_1(c^{-1}h_i, h_i)$, so it follows that the branch point of Δ_2 will be the same as that of the triangle $\{G_1, cG_1, c^{-1}G_1\}$. It follows that the branch point of Δ_2 is on the segment $[G_1, c^{-1}G_1]$ at distance $c_1(c, c^{-1})$ from the point G_1 . Therefore, $d_1(1, c^{-1}) = l_1(c) = c_1(c, c^{-1}) + c_1(c, h_i)$ \square

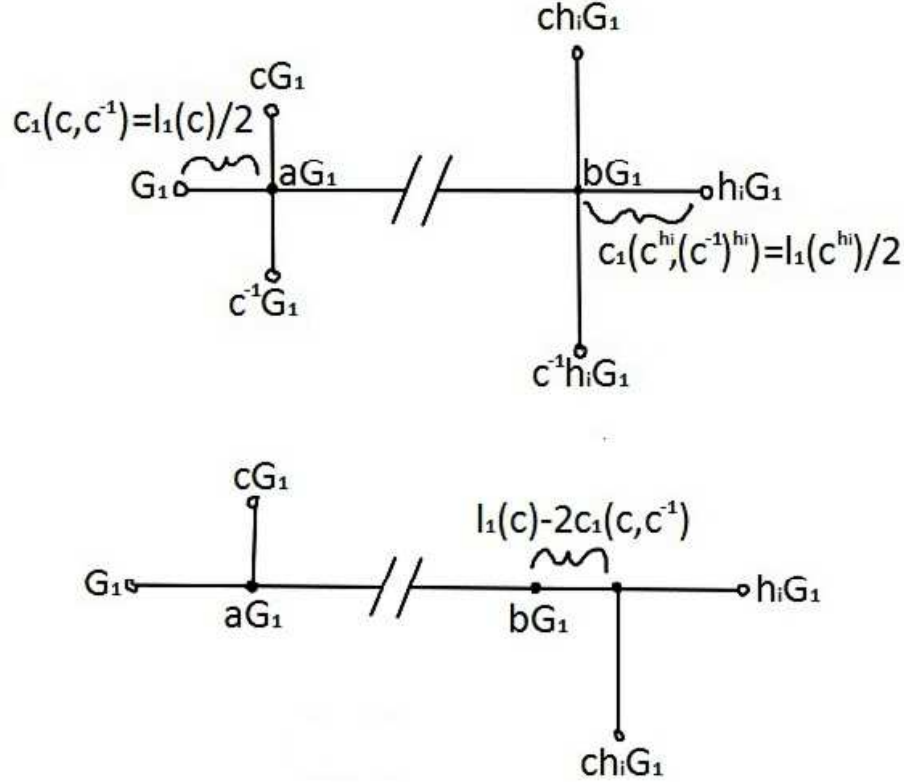
Definition 3.2.2 ([3]). Let G be a group acting on a Λ -tree X and $\ell(g) = \min\{d(x, gx) | x \in X\}$, we say the action is abelian if $\ell(gh) \leq \ell(g) + \ell(h)$ for any $g, h \in G$.

Proposition 3.2.2. Then the action of C_i on Γ_1^n is abelian.

Proof. For $c \in C_i$, either c is elliptic or hyperbolic. We can assume all mid-points and such are labeled vertices since the length function is δ -regular. The top figure details the elliptic case and the bottom one the hyperbolic case:

If c is elliptic, let aG_1 be the mid-point of $\{G_1, cG_1, c^{-1}G_1\}$ and bG_1 be the mid-point of $\{aG_1, ch_iG_1, h_iG_1\}$. We know aG_1 is fixed by c from [3, Lemma 3.1.1]. We also know aG_1 is co-linear with h_iG_1 since by the previous lemma we have $l_1(c) = c_1(c, h_i) + c_1(c, c^{-1})$ but we know $c_1(c, c^{-1}) = \frac{l_1(c)}{2}$ so $c_1(c, c^{-1}) = c_1(c, h_i)$. By extension bG_1 is fixed by c since $ht(d(aG_1, h_iG_1)) > ht(d(h_iG_1, ch_iG_1))$, so $[aG_1, bG_1] \subseteq A_c$.

Let c be hyperbolic and suppose without loss of generality that $l_1(c) = c_1(c, h_i) + c_1(c, c^{-1})$, we have that $\ell(c)$ is $l_1(c) - 2c_1(c, c^{-1})$ (see [3, Theorem 3.1.4] for proof of this and interpretation of $\ell(g)$ when g is hyperbolic). Let aG_1 be the mid-point of $\{G_1, cG_1, h_iG_1\}$, it is easy to see that, aG_1 being on both $[G_1, h_iG_1]$ and $[cG_1, ch_iG_1]$ (since $c_1(c, h_i) < c_1(ch_i, h_i)$) and $d_1(c, a) = c_1(c, c^{-1})$, we have



that $d_1(a, ca) = l_1(c) - 2c_1(c, c^{-1})$ (acG_1 is on $[cG_1, ch_iG_1]$ at distance $c_1(c, c^{-1})$ from cG_1).

Likewise, taking bG_1 to be the vertex on $[aG_1, h_iG_1]$ at distance $l_1(c) - 2c_1(c, c^{-1})$ from the mid-point of $\{G_1, h_iG_1, ch_iG_1\}$, a similar argument shows that $d_1(b, cb) = l_1(c) - 2c_1(c, c^{-1})$, so $[aG_1, bG_1] \subseteq A_c$.

Since in all cases, we have that A_c contains a sub-segment of $[G_1, h_iG_1]$, whose length has height $k+1$ whose left- and right-hand points are at distances from G_1 and h_iG_1 respectively of height at most k , we have that

$$\text{diam}(A_{c_1} \cap A_{c_2}) > l_1(c_1) + l_1(c_2)$$

for any $c_1, c_2 \in C_i$.

By a known result (see [3, Corollary 3.2.4]), this implies the action of C_i on Γ_1^n will be abelian or dihedral. If the action were dihedral, there would be a C_i -stable linear subtree on which C_i would act with at least one reflection. However, since $l_1(g^2) \geq l_1(g)$ for all g , there cannot be reflections. It follows the action is abelian. \square

Corollary 3.2.2. *Let E_i be the set of elements of C_i acting elliptically on Γ_1^n . Then $E_i \leq C_i$ and C_i/E_i will be free abelian of rank at most $k-1$.*

Proof. By known result (see [3, Proposition 3.2.7]), since the action of C_i on Γ_1^n is abelian, the hyperbolic length function l_1^h defined by $l_1^h(g) = l_1(g^2) - l_1(g)$ is a homomorphism towards \mathbb{Z}^{k-1} (the restriction of l_1 to $C_i \subseteq G_k$ takes values in \mathbb{Z}^{k-1}) whose kernel consists of elements whose action is elliptic. The corollary follows. \square

3.3 Special case where l is locally finite on G_1

This section consists of the proof of our main theorem, which follows directly from Proposition 3.3.1 and Proposition 3.2.1. The assumptions on G are the same as those of Theorem 3.1.1. We shall re-use the notation introduced in previous sections.

Lemma 3.3.1. *G_1 is torsion-free hyperbolic and quasi-isometric to (G_1, d_l) .*

Proof. $l|_{G_1}$ can be thought of as a length function in \mathbb{Z} since all but the left-most coordinates will be 0. We know it will be hyperbolic and regular, both of those properties will remain true since $ht(\delta) = 1$.

Consider the action of G_1 on $\Gamma_1(G_1, d_1)$ (see [7] for the definition of Γ_1 of a metric space). Since G_1 will be finitely generated by assumption and it is locally finite, it will have property $P(n)$ for any n large enough to contain a generating set. The lemma follows naturally from [7, Theorem 9]. \square

Lemma 3.3.2. *For any $g \in G$ and $n \in \mathbb{Z}(n \neq 0)$, if $l(g^n) \leq l(g)$, then $ht(l(g) - l(g^n)) = 1$.*

Proof. Let's look at the action of g on Γ_1^n . We know any isometry of a \mathbb{Z}^{n-1} -tree will be either elliptic, hyperbolic or an inversion. We will treat each of the three cases separately.

If the action of g is hyperbolic, then by [3, Theorem 1.4, Lemma 1.7, and Lemma 1.8] we have that $l_1(g^n) > l_1(g)$ for any $|n| > 1$.

If the action of g is an inversion, then there exists hG_1 such that $ghG_1 \neq hG_1$ but $g^2hG_1 = hG_1$. This implies that $g^h \notin G_1$ but $(g^h)^2 \in G_1$, so G_1 is not a primitive subgroup, contradicting our assumptions.

Finally, suppose g acts elliptically on the tree of cosets. Using the same argument as above, we know $Fix(g^n) = Fix(g)$ for any $n \neq 0$ since otherwise G_1 wouldn't be primitive. By [3, Lemma 1.1] we know that, for any $n \neq 0$,

$$\begin{aligned} d(G_1, g^n G_1) &= 2d(G_1, Fix(g^n)) = 2d(G_1, Fix(g)) = d(G_1, gG_1) \\ &\Rightarrow l_1(g^n) = l_1(g) \end{aligned}$$

\square

Lemma 3.3.3. *Suppose $h \notin G_k$. Then $Stab(hG_k) \cap G_1$ is either trivial or cyclic.*

Proof. Suppose $Stab(hG_k \cap G_1)$ is not trivial. Since we have that any $g \in Stab(hG_k) \cap G_1$ preserves hG_k , we have that $c(h, gh)$ has height equal to the height of h . It follows that

$$c(g, h) \geq \min\{c(g, gh), c(h, gh)\} - \delta = c(g, gh) - \delta = l(g) - c(g^{-1}, h) - \delta$$

It follows that either $c(g, h) \geq l(g)/2 - \delta$ or $c(g^{-1}, h) \geq l(g)/2 - \delta$. Let $\{x_n\}$ and $\{y_n\}$ be sequences of elements in G_1 such that $\{c(x_m, h)\}$ and $\{c(y_m, h)\}$ go to infinity. We know at least one such sequence exists since we have assumed $Stab(hG_k) \cap G_1$ non-trivial, and thus infinite, and $\max\{c(x_m, h), c(x_m^{-1}, h)\} \geq (l(x_m))/2 - \delta$ for any sequence $\{x_n\} \subset Stab(hG_k) \cap G_1$ and $Stab(hG_k) \cap G_1$ is not finite, so the word length in X of elements of $Stab(hG_k) \cap G_1$ cannot be bounded (we will use without reminder that $l(x) \simeq (0, \dots, 0, l_X(x))$ and so $c(x, y) \simeq (0, \dots, 0, c_X(x, y))$). Note that since we have that $c(x_m, y_n) \geq \min\{c(x_m, h), c(y_n, h)\} - \delta$, and $\lim_m c(x_m, h) = \infty = \lim_m c(y_n, h)$, we automatically have that $\lim_{m, n \rightarrow \infty} c_X(x_m, y_n) = \infty$. By the same argument, we have that $c(x_m, x_n) \geq \min\{c(x_m, h), c(x_n, h)\} - \delta$, and so $\lim_{m, n \rightarrow \infty} c_X(x_m, x_n) = \infty$. It follows that there is one and exactly one point on the boundary of G_1 which is the equivalence class of sequences convergent at infinity such that its elements have indefinitely large common initial segments with h . Let us call this point α .

Let now $w \in Stab(hG_k) \cap G_1$ and $\{x_n\}$ such that $\lim_{n \rightarrow \infty} c(x_n, h) = \infty$. $c(wx_n, h) \geq \min\{c(wx_n, wh), c(wh, h)\} - \delta$. Since $w \in Stab(hG_k) \cap G_1$, we have that

$$ht(h^{-1}wh) \leq k \Rightarrow ht(c(wh, h)) = ht(l(wh) + l(h) - l(h^{-1}wh)) = k+1 > ht(wx_n)$$

so we have that $c(wx_n, wh) < c(wh, h)$ and $c(wx_n, h) \geq c(wx_n, wh) - \delta$.

$$\begin{aligned} c(wx_n, wh) &= (l(wx_n) + l(wh) - l(x_n^{-1}h))/2 \\ &\geq (l(x_n) + l(h) - l(x_n^{-1}h) - 2l(w))/2 = c(x_n, h) - l(w) \end{aligned}$$

It follows that $\lim_{n \rightarrow \infty} c(wx_n, h) \geq \lim_{n \rightarrow \infty} c(x_n, h) - l(w) = \infty$. It follows that $Stab(hG_k) \cap G_1$ is a subgroup of the stabilizer of α . We know ([5, Theorem 8.30]) that $Stab_{G_1}(\alpha)$ is virtually cyclic, and since G is torsion-free, so is $Stab_{G_1}(\alpha)$, so by known result ([11, Lemma 3.2]), $Stab_{G_1}(\alpha)$ must be cyclic. It follows that so must be $Stab(hG_k) \cap G_1$. \square

Let us recall a definition given in Corollary 3.2.2. If C_i consists of the elements of G_k which stabilize h_iG_k , then we define E_i to be the set of elements of C_i acting elliptically on Γ_1^n . We recall that this set is a normal subgroup of C_i and that C_i/E_i is free abelian of rank at most $k-1$.

Lemma 3.3.4. *Letting again E_i be the set of all elements of C_i that act elliptically on Γ_1^n . E_i is either trivial or infinite cyclic.*

Proof. By Lemma 3.2.3, since any $g \in E_i$ acts elliptically on Γ_1^n , we have that $c_1(g, g^{-1}) = l_1(g)/2 = c_1(g, h_i)$, which implies that, for any $g_1, g_2 \in E_i$, $c_1(g_1^{-1}, g_2) \geq \min\{c_1(g_1, h_i), c_1(g_2, h_i)\} = \min\{l_1(g_1), l_1(g_2)\}/2$, so $l_1(g_1 g_2) \leq \max\{l_1(g_1), l_1(g_2)\}$, so for any $g \in E_i$, $E_i(g) = \{h \in E_i | l_1(h) \leq l_1(g)\}$ is a subgroup of E_i .

For any $g \in E_i$, let $c_g G_1$ be the mid-point of $\{G_1, gG_1, h_i G_1\}$. Since $d(G_1, c_g G_1) = c_1(g, h_i) = l(g)/2 = d(G_1, gG_1)/2$, we have that $c_g G_1$ is stabilized by g . Let $d_g G_1$ be the mid-point of $\{c_g G_1, gh_i G_1, h_i G_1\}$. Since $c_g G_1$ is in the axis of g , we have that so is $d_g G_1$. Furthermore, since $c_g G_1 \in [G_1, h_i G_1]$, we have that $d_g G_1$ is also the mid-point of $\{G_1, gh_i G_1, h_i G_1\}$, and so is found on $[G_1, h_i G_1]$ and $d(G_1, d_g G_1) = c_1(h_i, gh_i)$.

Suppose then that $h \in E_i(g)$, in other words $h \in E_i$ and $l_1(h) \leq l_1(g)$. By the above we have that $[c_g G_1, d_g G_1] \subseteq [c_h G_1, d_h G_1]$ which means h stabilizes $c_g G_1$.

We thus have that $c_g^{-1} E_i(g) c_g \leq G_1$. Furthermore, since $E_i(g)$ stabilizes $h_i G_k$, we have that $c_g^{-1} E_i(g) c_g$ stabilizes $c_g^{-1} h_i G_k$, and since $c_g \in G_k$ and $h_i \notin G_k$, we have that $c_g^{-1} h_i G_k \neq G_k$. Suppose $E_i(g)$ is not trivial. By Lemma 3.3.3, since $c_g^{-1} E_i(g) c_g \leq G_1$ preserves $c_g^{-1} h_i G_k$, we have that $c_g^{-1} E_i(g) c_g$ must be cyclic, and thus $E_i(g)$ must be cyclic.

Suppose now there exist non-trivial $g, h \in E_i$ such that $l_1(g) < l_1(h)$. We know both $E_i(g)$ and $E_i(h)$ are cyclic, and we know $E_i(g) < E_i(h)$, so it follows that $[E_i(h) : E_i(g)]$ is finite. However, that implies that there exist different m, n such that $h^m E_i(g) = h^n E_i(g)$, in other words that $h^{m-n} \in E_i(g)$, thus that $l_1(h^{m-n}) < l_1(h)$, which is impossible.

It follows that all non-trivial elements of E_i have the same l_1 , thus that E_i is either trivial or $E_i = E_i(g)$ for some g , meaning that E_i is cyclic. \square

Lemma 3.3.5. *The edge groups of the hierarchy are nilpotent of rank at most 3.*

Proof. We know by Lemma 3.3.4 that the subgroup of elliptic elements of C_i is cyclic. We also know by Lemma 3.2.2 that $E_i \trianglelefteq C_i$ and C_i/E_i is free abelian of rank at most $k-1$. Since E_i is normal, there exists a homomorphism $\varphi : C_i \rightarrow \text{Aut}(\mathbb{Z}) = \mathbb{Z}_2$ defined by $h^{\varphi(g)} = h^g$ for any $h \in E_i, g \in C_i$. Let H_i be the kernel of that homomorphism. We know H_i/E_i is abelian and $E_i \subseteq Z(H_i)$, so we have that $H_i/Z(H_i)$ is abelian.

This implies that the pull-back of $Z(H_i/Z(H_i)) = H_i$, so H_i is nilpotent of rank at most 2. Finally, since $H_i \trianglelefteq C_i$ and C_i/H_i is either 1 or \mathbb{Z}_2 , both of which are abelian, we also have C_i has to be nilpotent of rank at most 3. \square

Proposition 3.3.1. *For any k , G_{k+1} is an HNN extension of G_k with a finite number of stable letters and the associated subgroups are virtually free abelian of rank at most k .*

Proof. All that remains to be proven is that G_{k+1} is a finite HNN extension of G_k .

By Corollary 3.2.1, it suffices to prove that G_j is finitely generated for any j . We know $G_n = G$ is finitely generated, and so a finite HNN extension of G_{n-1} . Let $\{C_1, \dots, C_k\}$ be the subgroups associated used in the construction. We know C_i is finitely generated for any i , so let $C_i = \langle c_{i,1}, \dots, c_{i,j_i} \rangle$ and let $h_i^{-1}c_{i,k}h_i = d_{i,k}$. It follows that $G_n = \langle G_{n-1}, h_1, \dots, h_k | h_i^{-1}c_{i,k}h_i = d_{i,k} \rangle$. G_n is then finitely generated and finitely presented relatively to G_{n-1} . It follows by known result ([14, Theorem 1.1]) that G_{n-1} is itself finitely generated. Repeating this argument for G_{n-1} and G_{n-2} , then for G_{n-2} and G_{n-3} and so on... gives us that G_j is finitely generated for any j , so in particular G_{k+1} is a finite HNN extension of G_k . \square

The main result, Theorem 3.1.1, follows easily from the above proposition.

Corollary 3.3.1. *If $n = 2$, then G is relatively hyperbolic with abelian parabolics. If $n = 3$, then all edge groups are either cyclic or virtually free abelian of rank 2.*

Proof. By the above results, if $n = 2$, then G is an HNN extension of a hyperbolic group with a finite number of stable letters and associated cyclic subgroups. Furthermore, since those subgroups are cyclic, they must be maximal. Indeed, the associated subgroups are of the form $h_i G_1 h_i^{-1} \cap G_1$ for some h_i ; if they were properly contained in some larger cyclic subgroup, say $h_i G_1 h_i^{-1} \subset \langle c \rangle$, then there would be some $n \in \mathbb{Z}$ such that $l_1(h_i c h_i^{-1}) \neq 0$ but $l_1(h_i c^n h_i^{-1}) = 0$, which is a contradiction. Since G is a finite HNN extension of a hyperbolic group with maximal cyclic subgroups, the corollary follows by known result ([4, Theorem 0.1]).

If $n = 3$, then the edge groups will either be cyclic or cyclic-by-cyclic. The result follows easily. \square

References

- [1] R. Alperin and H. Bass. Length functions of group actions on Λ -trees. In *Combinatorial group theory and topology*, ed. S. M. Gersten and J. R. Stallings, volume 111 of *Annals of Math. Studies*, pages 265–178. Princeton University Press, 1987.
- [2] I. Chiswell. Abstract length functions in groups. *Math. Proc. Cambridge Philos. Soc.*, 80(3):451–463, 1976.
- [3] I. Chiswell. *Introduction to Λ -trees*. World Scientific, 2001.
- [4] F. Dahmani. Combination of convergence groups. *Geom. Topol.*, 7:933–963, 2003.
- [5] E. Ghys and P. de la Harpe. *Espaces Métriques Hyperboliques sur les Groupes Hyperboliques d’après Michael Gromov*. Birkhauser, 1991.
- [6] A.-P. Grecianu. *Group Actions on Non-Archimedean Hyperbolic Spaces*. PhD thesis, McGill University, 2013.

- [7] A.-P. Grecianu, A. Kvaschuk, A. G. Myasnikov, and D. Serbin. Groups acting on hyperbolic Λ -metric spaces. *Internat. J. Algebra Comput.*, 25(6):977–1042, 2015.
- [8] O. Kharlampovich, A. G. Myasnikov, V. N. Remeslennikov, and D. Serbin. Groups with free regular length functions in \mathbb{Z}^n . *Trans. Amer. Math. Soc.*, 364:2847–2882, 2012.
- [9] O. Kharlampovich, A. G. Myasnikov, and D. Serbin. Actions, length functions, and non-archimedean words. *Internat. J. Algebra Comput.*, 23(2):325–455, 2013.
- [10] R. Lyndon. Length functions in groups. *Math. Scand.*, 12:209–234, 1963.
- [11] D. Macpherson. Permutation groups whose subgroups have just finitely many orbits. In *Ordered groups and infinite permutation groups*, volume 354 of *Mathematics and Its Applications*, pages 221–229. Springer, 1996.
- [12] J. Morgan and P. Shalen. Valuations, trees, and degenerations of hyperbolic structures, I. *Ann. of Math.*, 120:401–476, 1984.
- [13] A. G. Myasnikov, V. Remeslennikov, and D. Serbin. Regular free length functions on Lyndon’s free $\mathbb{Z}[t]$ -group $F^{\mathbb{Z}[t]}$. In *Algorithms, Languages, Logic*, volume 378 of *Contemporary Mathematics*, pages 37–77. American Mathematical Society, 2005.
- [14] D. Osin. *Relatively hyperbolic groups: Intrinsic geometry, algebraic properties, and algorithmic problems*, volume 179 of *Mem. Amer. Math. Soc.* AMS Press, Providence, 2006.
- [15] D. Promislow. Equivalence classes of length functions on groups. *Proc. London Math. Soc.*, 51(3):449–477, 1985.
- [16] J.-P. Serre. *Trees*. New York, Springer, 1980.
- [17] D. T. Wise. The structure of groups with a quasiconvex hierarchy. Preprint. Available at <https://docs.google.com/file/d/0B45cNx80t5-2NTU0ZTdhMmItZTlxOS00ZGUyLWE0YzItNTEyYWFiMjczZmIz/edit?pli=1>.